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EXPANSIONS OF THE NEUMANN TYPE IN TERMS OF PRODUCTS OF BESSEL FUNCTIONS

BY

GUY STEVENSON

A. B., Georgetown College, 1921,
A. M., University of Illinois, 1923.

2 P.

Thesis

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS IN THE GRADUATE SCHOOL OF THE UNIVERSITY OF ILLINOIS, 1926

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Expansions of the Neumann Type in Terms of Products of Bessel Functions.

By Guy Stevenson.

I. Introduction.

1. *Statement of the Problem.* The object of this paper is the generalization of certain expansions of functions in terms of Bessel functions, and related functions, which expansions have been investigated by Neumann, Gegenbauer and others. In his *Theorie der Bessel'schen Funktionen* Neumann has obtained the expansion *

$$f(z) = \sum_{n=0}^{\infty} a_n J_n(z),$$

where $f(z)$ is single-valued and analytic in the neighborhood of the origin, and the coefficients a_n are constants whose values are reproduced in Art. 13. This expansion has been generalized by Gegenbauer † to one of the form ‡

$$z^{\nu} f(z) = \sum_{n=0}^{\infty} b_n J_{\nu+n}(z),$$

where ν is any complex quantity, except a negative integer, and the coefficients b_n are constants whose values are reproduced in Art. 13. Also, to Gegenbauer Watson attributes the expansion §

$$z^{\mu+\nu} f(z) = \sum_{n=0}^{\infty} c_n J_{\mu+n/2}(z) J_{\nu+n/2}(z),$$

where μ and ν are any complex quantities, except the halves of negative integers, and the coefficients c_n are constants whose values are also given in Art. 13.

In this paper we extend the expansion theories of Neumann and Gegenbauer in four general directions. In the first place each term in the generalized expansions contains, as factors, any number n of Bessel functions. In

* See also Watson, *Theory of Bessel Functions* (London, 1922), p. 523.

† Wiener Sitzungsberichte, 74 (2), (1877), pp. 124-130; Watson, *loc cit.*, p. 524.

‡ By an expression of the form a^r we shall uniformly mean $e^{r \log a}$ where the principal determination of $\log a$ is taken.

§ Wiener Sitzungsberichte, 75, (2), (1877), pp. 218-222; Watson, *loc. cit.*, p. 525.

the second place the order of each of the Bessel functions in the expansions has the more general form $\mu_i + \theta_i m$ where the μ_i are constants, real or complex, the θ_i are certain positive constants and m is a non-negative integer. In the third place the argument of the Bessel functions is extended in an obvious manner, from z to $z - a$, where a is any real or complex quantity. And, finally, the extension is made to the expansion of functions of several variables.

We first obtain the expansion

$$(z/2)^{\alpha+k} = \sum_{m=0}^{\infty} c_{k,m}\, \mathfrak{F}_{k+2m}(z),$$

in which $\alpha = \mu_1 + \mu_2 + \cdots + \mu_n$, k is a non-negative integer, the coefficients $c_{k,m}$ are constants which are defined in Art. 2, and

$$\mathfrak{F}_m(z) = J_{\mu_1+\theta_1 m}(z) J_{\mu_2+\theta_2 m}(z) \cdots J_{\mu_n+\theta_n m}(z),$$

always subject to the following restrictions: none of the quantities $\mu_i + \theta_i m$ shall be a negative integer, for $i = 1, 2, \cdots, n$, and for $m = 0, 1, 2, \cdots$, and $\theta_1 + \theta_2 + \cdots + \theta_n$ shall be 1 or 2. Since the case presenting the most interest is that for which the sum of the θ's is 1 we make this assumption unless otherwise specified. We next develop an expansion of the function $f(z)$ which is analytic and single-valued within and upon the circle C defined by the relation $|z - a| = R$ in the form

$$f(z) = \sum_{m=0}^{\infty} a_m z^{-\alpha}\, \mathfrak{F}_m(z - a)$$

where

$$a_m = (1/2\pi i) \int_C B_m(t - a)\, f(t)\, dt,$$

and $B_m(t - a)$ is a polynomial in $1/(t - a)$ whose exact form is given in Art. 6. We then obtain expansions for functions in terms of the polynomials $B_m(z - a)$.

In the remaining sections a relation between these expansions and those of Taylor and Laurent is obtained, certain orthogonality properties are derived, some special cases are considered, expansions for the case for which the sum of the θ's is 2 are obtained, and, finally, the extension is made to the expansion of functions of several variables.

II. Expansion of $(z/2)^{a+k}$ in Terms of the Functions $\mathfrak{F}_{k+2m}(z)$.

2. *Expansion of* $\mathfrak{F}_m(z)$ *in powers of* $(z/2)$. If the absolutely convergent series which represent the Bessel functions $J_{\mu_i+\theta_i m}(z)$, namely,

$$(1) \qquad J_{\mu_i+\theta_i m}(z) = \sum_{r=0}^{\infty} (-1)^r (z/2)^{\mu_i+\theta_i m+2r}/r!\,\Gamma(\mu_i+\theta_i m+r+1), \qquad (i=1, 2, \cdots, n),$$

be multiplied together and the product arranged according to the powers of $z/2$ the coefficient of $(z/2)^{a+m+2s}$ is $(-1)^s a_{m,s}$, where

$$a_{m,s} = \sum_{i_1, i_2, \ldots, i_n=0}^{s} 1/i_1!\, i_2! \cdots i_n!\, \Gamma(\mu_1+\theta_1 m+i_1+1)\, \Gamma(\mu_2+\theta_2 m+i_2+1) \cdots \Gamma(\mu_n+\theta_n m+i_n+1),$$

the summation being taken over all non-negative integral solutions of the equation $i_1+i_2+\cdots+i_n=s$.

Since the series (1), which define the Bessel functions, all converge absolutely and uniformly in any closed region of the z-plane (the origin being excepted if the real part of $\mu_i+\theta_i m$ is negative) it follows that the expansion

$$(2) \qquad \mathfrak{F}_m(z) = \sum_{s=0}^{\infty} (-1)^s a_{m,s} (z/2)^{a+m+2s}$$

is also absolutely and uniformly convergent in the same region.

3. *Formal expansion of* $(z/2)^{a+k}$ *in terms of the functions* $\mathfrak{F}_{k+2m}(z)$. In order to determine the coefficients $c_{k,m}$ in the expansion

$$(3) \qquad (z/2)^{a+k} = \sum_{m=0}^{\infty} c_{k,m}\, \mathfrak{F}_{k+2m}(z), \qquad (k=0, 1, 2, 3, \cdots),$$

which expansion is momentarily assumed, we substitute for each of the functions $\mathfrak{F}_{k+2m}(z)$ the appropriate power series expansion obtained from (2) and equate coefficients of like powers of $(z/2)$. This gives the recurrence relations

$$(4) \qquad \begin{cases} a_{k,0}c_{k,0} & = 1, \\ a_{k,1}c_{k,0} - a_{k+2,0}c_{k,1} & = 0, \\ a_{k,2}c_{k,0} - a_{k+2,1}c_{k,1} + a_{k+4,0}c_{k,2} & = 0, \\ \cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots & \\ a_{k,m}c_{k,0} - a_{k+2,m-1}c_{k,1} + \cdots + (-1)^m a_{k+2m,0}c_{k,m} & = 0, \end{cases}$$

whence we have

$$(5)\quad c_{k,m} = \frac{\begin{vmatrix} a_{k,1} & a_{k+2,0} & 0 & 0 \cdots 0 \\ a_{k,2} & a_{k+2,1} & a_{k+4,0} & 0 \cdots 0 \\ \cdots & \cdots & \cdots & \cdots \\ a_{k,m-1} & a_{k+2,m-2} & a_{k+4,m-3} \cdots & a_{k+2m-2,0} \\ a_{k,m} & a_{k+2,m-1} & a_{k+4,m-2} \cdots & a_{k+2m-2,1} \end{vmatrix}}{a_{k,0}a_{k+2,0}\cdots a_{k+2m,0}}$$

$$= \frac{\begin{vmatrix} a'_{k,1} & 1 & 0 & 0 \cdots 0 \\ a'_{k,2} & a'_{k+2,1} & 1 & 0 \cdots 0 \\ \cdots & \cdots & \cdots & \cdots \\ a'_{k,m-1} & a'_{k+2,m-2} & a'_{k+4,m-3} \cdots & 1 \\ a'_{k,m} & a'_{k+2,m-1} & a'_{k+4,m-2} \cdots & a'_{k+2m-2,1} \end{vmatrix}}{a_{k+2m,0}}$$

where

$$a'_{r,s} = a_{r,s}/a_{r,0} = \sum_{i_1, i_2, \ldots, i_n=0}^{s} 1/\, i_1!\, i_2! \cdots i_n!\, (\mu_1 + \theta_1 r + i_1)_{i_1} (\mu_2 + \theta_2 r + i_2)_{i_1} \cdots (\mu_n + \theta_n r + i_n)_{i_n}$$

and $$(a)_s = a(a-1)(a-2)\cdots(a-s+1).$$

If we denote by $\Delta_{k,m}(a')$ the determinant in the last member of (5) we have

$$(6)\quad c_{k,m} = \Delta_{k,m}(a')/a_{k+2m,0}.$$

In the next sections we show that the series

$$(7)\quad \sum_{m=0}^{\infty} c_{k,m}\,(z/2)^{-a}\,\mathfrak{F}_{k+2m}(z)$$

is absolutely and uniformly convergent in any closed region and that the series in (3) represents the function $(z/2)^{a+k}$.

4. *Expressions dominating $a_{m,s}$, $c_{k,m}$ and $\mathfrak{F}_m(z)$.* We first obtain simple dominants for the quantities $a_{m,s}$, $c_{k,m}$ and $\mathfrak{F}_m(z)$, which will be used later in the convergence proofs. From the definition of $a_{m,s}$ (Art. 2) we see that

$$(8)\quad |\, a_{m,s}\,| = \Big|\sum_{i_1, i_2, \ldots, i_n=0}^{s} 1/i_1!\, i_2! \cdots i_n!\, \Gamma(\mu_1+\theta_1 m+i_1+1) \times \Gamma(\mu_2+\theta_2 m+i_2+1)\cdots\Gamma(\mu_n+\theta_n m+i_n+1)\Big|$$

$$\leq \sum_{i_1, i_2, \ldots, i_n=0}^{s} \frac{s!/(i_1!\, i_2! \cdots i_n!)\;|\,(\mu_1+\theta_1 m+s)_{s-i_1}(\mu_2+\theta_2 m+s)_{s-i_2}\cdots(\mu_n+\theta_n m+s)_{s-i_n}\,|}{s!\,|\,\Gamma(\mu_1+\theta_1 m+s+1)\,\Gamma(\mu_2+\theta_2 m+s+1)\cdots\Gamma(\mu_n+\theta_n m+s+1)\,|}$$

where the summation is taken over all non-negative integral solutions of the equation $i_1 + i_2 + \cdots + i_n = s$. If now we allow the summation to be taken over all non-negative integral values consistent with the relation $i_1 + i_2 + \cdots + i_n = s(n-1)$ then the last member of (8) will not be decreased.* That is, we have

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* The truth of this statement is established in the following manner:

By a comparison of the terms entering into the sum in the two cases for which $i_1 + i_2 + \cdots + i_n$ is s and $s(n-1)$ respectively, we observe that to each term in the first case there corresponds a term in the second case which differs from it only in the coefficients involving the factorials. Consequently, it is sufficient to show that, for each set of the i's consistent with the relation $i_1 + i_2 + \cdots + i_n = s$ we have

$$s!/i_1!\, i_2! \cdots i_n! \leq [s(n-1)]!/(s-i_1)!\,(s-i_2)! \cdots (s-i_n)!.$$

This is true if

$$s!/[s(n-1)]! \leq i_1!\, i_2! \cdots i_n!/(s-i_1)!\,(s-i_2)! \cdots (s-i_n)!.$$

Denote by $f(s, n;\ i_1, i_2, \cdots, i_n)$ the second member of the last inequality. The last inequality is obviously true for $n = 1$ or 2 if s is any non-negative integer. It is also clearly true when $s = 0,\ 1,\ 2$ for all values of n greater than 2. Therefore, in what follows, we may assume that s and n are each greater than 2. Let $p_1,\ p_2,\ p_3, \cdots, p_n$ denote a set of values which the i's may assume that will make the product $i_1!\, i_2! \cdots i_n!$ a minimum. Then we have

$$\begin{aligned} s!/i_1!\, i_2! \cdots i_n! &\leq s!/p_1!\, p_2! \cdots p_n! \\ &\leq [s(n-1)]!/(s-p_1)!\,(s-p_2)! \cdots (s-p_n)! \end{aligned}$$

for every possible set of the i's. Hence

$$\begin{aligned} s!/[s(n-1)]! &\leq p_1!\, p_2! \cdots p_n!/(s-p_1)!\,(s-p_2)! \cdots (s-p_n)! \\ &= f(s, n;\ p_1, p_2, \cdots, p_n). \end{aligned}$$

Now the proposition is established if it is shown that $f(s, n;\ p_1, p_2, \cdots, p_n)$ is no greater than $f(s, n;\ i_1, i_2, \cdots, i_n)$ for any set of values which the i's may take. If s and n are each greater than 2, and r is the integer satisfying the relation $r = s/n + \theta$ where $o \leqq \theta < 1$, then $f(s, n;\ i_1, i_2, \cdots, i_n)$ is seen to be simply related to $f(s, n;\ p_1, p_2, \cdots, p_n)$. In fact, after certain obvious reductions have been made, we have

$$\begin{aligned} f(s, n;\ i_1, i_2, \cdots, i_n) = f(s, n;\ p_1, p_2, \cdots, p_n)\,&[(r+a_1)(s-r-a_1+1)] \cdots \\ &[(r+a_m)(s-r-a_m+1)]/[(r-b_1)(s-r+b_1+1)] \cdots \\ &[(r-b_m)(s-r+b_m+1)], \end{aligned}$$

where a_i and b_i are suitably determined non-negative integers. Since no factor in brackets in the numerator of the fraction in the last relation can be less than $k(s-k+1)$ and no factor in brackets in the denominator can be greater than $k(s-k+1)$, it is clear that $f(s, n;\ i_1, i_2, \cdots, i_n) \geqq f(s, n;\ p_1, p_2, \cdots, p_n)$ for every possible set of i's. Hence the proposition is established.

$$(9)\quad |\, a_{m,s}\,| \leq \sum_{i_1,\, i_2,\, \ldots,\, i_n=0}^{s(n-1)} \frac{[s(n-1)]\,!/i_1\,!\,i_2\,!\cdots i_n\,!\;|\,(\mu_1+\theta_1 m+s)_{i_1}(\mu_2+\theta_2 m+s)_{i_2}\cdots(\mu_n+\theta_n m+s)_{i_n}\,|}{s\,!\,|\,\Gamma(\mu_1+\theta_1 m+s+1)\,\Gamma(\mu_2+\theta_2 m+s+1)\cdots\Gamma(\mu_n+\theta_n m+s+1)\,|}$$

$$= |\,\alpha+m+ns\,|_{s(n-1)}\,/s\,!\,|\,\Gamma(\mu_1+\theta_1 m+s+1)\times \Gamma(\mu_2+\theta_2 m+s+1)\cdots\Gamma(\mu_n+\theta_n m+s+1)\,|$$

$$\leq \Gamma(|\,\alpha\,|+m+ns+1)\,\Gamma(|\,\alpha\,|+m+s+1)\,|\,\Gamma(\mu_1+\theta_1 m+s+1)\times \Gamma(\mu_2+\theta_2 m+s+1)\cdots\Gamma(\mu_n+\theta_n m+s+1)\,|.$$

The summation indicated in the second member of (9) has been effected by means of the identity

$$(a_1+a_2+\cdots+a_n)_s = \sum_{i_1,\, i_2,\, \ldots,\, i_n=0}^{s} [s\,!/i_1\,!\,i_2\,!\cdots i_n\,!](a_1)_{i_1}(a_2)_{i_2}\cdots(a_n)_{i_n}$$

an identity which is readily established by mathematical induction. The special case for which $n=2$ was established by Vandermonde.* For convenience we shall denote by $\alpha_{m,s}$ the last member of (9).

From the recurrence relations obtained in Art. 2 we have the relation

$$(10)\quad |\,c_{k,m}\,| \leq [|\,a_{k,m}\,|\cdot|\,c_{k,0}\,|+|\,a_{k+2,m-1}\,|\cdot|\,c_{k,1}\,|+\cdots +|\,a_{k+2m-2,1}\,|\cdot|\,c_{k,m-1}\,|]/|\,a_{k+2m,0}\,|$$

$$\leq [\alpha_{k,m}\,|\,c_{k,0}\,|+\alpha_{k+2,m-1}\,|\,c_{k,1}\,|+\cdots +\alpha_{k+2m-2,1}\,|\,c_{k,m-1}\,|]/|\,a_{k+2m,0}\,|.$$

We shall denote the last member of (10) by $B_{k,m}$.

If we expand the determinant $\Delta_{k,m}(a')$ by minors with respect to the elements of the last column, and then expand in the same manner the minor of the unit element and so on, we have

$$(11)\quad \Delta_{k,m}(a') = a'_{k+2m-2,1}\Delta_{k,m-1}(a') - a'_{k+2m-4,2}\Delta_{k,m-2}(a') + \cdots + (-1)^m a'_{k+2,m-1}\Delta_{k,1}(a') + (-1)^{m+1}a'_{k,m}.$$

Now let $P_{k,m}(a')$ denote the sum of the absolute values of the terms in the expansion of $\Delta_{k,m}(a')$ as a polynomial in the symbols a'. Then

$$(12)\quad P_{k,m}(a') = |\,a'_{k,m}\,|+|\,a'_{k+2,m-1}\,|\,P_{k,1}(a')+|\,a'_{k+4,m-2}\,|\,P_{k,2}(a')+\cdots +|\,a'_{k+2m-2,1}\,|\,P_{k,m-1}(a').$$

If in (12) we replace $|\,a'_{r,s}\,|$ by the dominating quantity $a'_{r,s}$, where

$$a'_{r,s} = \Gamma(-|\,\alpha\,|+r+ns+1)/\,\Gamma(-|\,\alpha\,|+r+s+1)\,(-|\,\mu_1\,|+\theta_1 r+s)_s \times (-|\,\mu_2\,|+\theta_2 r+s)_s\cdots\ (-|\,\mu_n\,|+\theta_n r+s)_s,$$

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* See Chrystal's *Algebra*, Vol. 2, p. 9.

we have *

$$(13)\quad P_{k,m}(\alpha') = \alpha'_{k,m} + \alpha'_{k+2,m-1}P_{k,1}(\alpha') + \alpha'_{k+4,m-2}P_{k,2}(\alpha') + \cdots + \alpha'_{k+2m-2,1}P'_{k,m-1}(\alpha').$$

Moreover, we have

$$(14)\quad |\, c_{k,m}\,| \leq P_{k,m}(\alpha')/|\, a_{k+2m,0}\,| \leq P_{k,m}(\alpha')/|\, a_{k+2m,0}\,|.$$

Making use of the inequality employed by Nielsen,†

$$|\, J_\nu(z)\,| \leq |\, (z/2)^\nu\,|\, e^{|z|^2/4|\nu_0+1|}/\Gamma(\nu+1),$$

where $|\, \nu_0 + 1\,|$ is the least of the quantities $|\, \nu + 1\,|, |\, \nu + 2\,|, \cdots$, we have the following inequality:

$$(15)\quad |\, \mathfrak{F}_m(z)\,| \leq |\, (z/2)^{\alpha+m}\,| \cdot |\, a_{m,0}\,|\, e^{|z/2|^2 E_m},$$

where $E_m = \sum_{i=1}^{n} 1/\nu_i(m)$, and $\nu_i(m)$ for each i is the smallest of the quantities $|\, \mu_i + \theta_i m + r\,|$ for $r = 1, 2, 3, \cdots$.

5. *Convergence of the series in* (3). In view of the inequalities (10) and (15) it is clear that the series in (3) is dominated, term by term, by the series

$$(16)\quad \sum_{n=0}^{\infty} B_{k,m}\,|\, a_{k+2m,0}\,| \cdot |\, (z/2)^{\alpha+k+2m}\,|\, e^{|z/2|^2 E_{k+2m}}.$$

Denote by U_m the mth term of (16), and consider the limit of the ratio U_{m+1}/U_m as m becomes infinite. From (16) and (10) we have

$$(17)\quad U_{m+1}/U_m = [B_{k,m+1}\,|\, a_{k+2m+2,0}\,|/B_{k,m}\,|\, a_{k+2m,0}\,|]\,|\, z/2\,|^2 \exp[\,|\, z/2\,|^2\{E_{k+2m+2} - E_{k+2m}\}]$$

$$\leq \left[\frac{\alpha_{k,m+1}\,|\, c_{k,0}\,| + \alpha_{k+2,m}\,|\, c_{k,1}\,| + \cdots + \alpha_{k+2m-2,2}\,|\, c_{k,m-1}\,|}{\alpha_{k,m}\,|\, c_{k,0}\,| + \alpha_{k+2,m-1}\,|\, c_{k,1}\,| + \cdots + \alpha_{k+2m-2,1}\,|\, c_{k,m-1}\,|} + \frac{\alpha_{k+2m,1}}{|\, a_{k+2m,0}\,|}\right] |\, z/2\,|^2 \exp[\,|\, z/2\,|^2\{E_{k+2m+2} - E_{k+2m}\}].$$

Consider the limit $L_{m\to\infty} f(m,s)$ where $f(m,s) = \alpha_{k+2s,\,m+1-s}/\alpha_{k+2s,\,m-s}$, in which s may take non-negative integral values subject to the restriction that it be not greater than m. From the definition of $\alpha_{r,s}$ (Art. 4) we see that

$$f(m,s) = \frac{[|\alpha|+k+2s+n(m-s)+1][|\alpha|+k+2s+n(m-s)+2]\cdots[|\alpha|+k+2s+n(m-s)+n]}{(m-s+1)\,[\,|\alpha|+k+s+m+1\,]\,[\mu_1+\theta_1(k+2s)+m-s+1\,]\cdots[\mu_n+\theta_n(k+2s)+m-s+1\,]}.$$

* We note that if $k = -|\, u_i\,|/\theta_i$ for any i of the set $i = 1, 2, \cdots, n$, the quantities $P_{k,m}(\alpha')$ are undefined. Hence we exclude all such values of k.

† *Mathematischen Annalen*, Vol. 52, (1899) p. 231; see Watson, *loc. cit.*, p. 44.

We observe that the factors of the numerator may be paired with the last n factors of the denominator so that each of the resulting n ratios is bounded as to m and s. Let M denote the product of these n bounds. The limit $\underset{m\to\infty}{L} f(m,s)$ is then seen to be zero uniformly as to s when s is in the range $0 \leq s \leq m$.

Then, for every positive N, there exists an integer m' such that for all values of $m > m'$ and for all values of s in the range $0 \leq s \leq m$, we have

$$\alpha_{k+2s,\,m+1-s}\, N < \alpha_{k+2s,m-s}. \tag{18}$$

Then for all values of $m > m'$ we have

$$\frac{U_{m+1}}{U_m} \leq \frac{1}{N}$$

$$\left[\frac{N\alpha_{k,m+1}\,|c_{k,0}| + N\alpha_{k+2,m}\,|c_{k,1}| + \cdots + N\alpha_{k+2m-2,2}\,|c_{k,m-1}|}{\alpha_{k,m}\,|c_{k,0}| + \alpha_{k+2,m-1}\,|c_{k,1}| + \cdots + \alpha_{k+2m-2,1}\,|c_{k,m-1}|} + \frac{N\alpha_{k+2m,1}}{|a_{k+2m,0}|}\right] R_m(z)$$

$$\leq \frac{1}{N}$$

$$\left[\frac{\alpha_{k,m}\,|c_{k,0}| + \alpha_{k+2,m-1}\,|c_{k,1}| + \cdots + \alpha_{k+2m-2,1}\,|c_{k,m-1}|}{\alpha_{k,m}\,|c_{k,0}| + \alpha_{k+2,m-1}\,|c_{k,1}| + \cdots + \alpha_{k+2m-2,1}\,|c_{k,m-1}|} + \frac{\alpha_{k+2m,1}\,N}{|a_{k+2m,0}|}\right] R_m(z)$$

$$= \frac{1}{N}\left[1 + \frac{N\,\alpha_{k+2m,1}}{|a_{k-2m,0}|}\right] R_m(z),$$

where $R_m(z) = |z/2|^2 \exp\left[|z/2|^2 \{E_{k+2m+2} - E_{k+2m}\}\right]$,

and, since $\underset{m\to\infty}{L} \alpha_{k+2m,1}/|a_{k+2m,0}| = 0$, and $\underset{m\to\infty}{L} E_m = 0$, we see that

$$\underset{m\to\infty}{L} U_{m+1}/U_m \leq |z/2|^2/N < 1 \qquad \text{if} \qquad |z/2|^2 < N.$$

Hence the series (16) and consequently the series in (3) converges absolutely for all finite values of z.

The uniformity of convergence of the series (7) throughout any closed region is readily established. For, let R' be any constant, and, let $R > R'$. Then, if in (16) each term be divided by $(z/2)^a$ and z be replaced by R, we have a convergent series of positive constants by which the series in (7) is dominated term by term provided that $|z| \leq R'$. Hence the uniformity of convergence throughout the given region follows from the test of Weierstrass.

Since the power series expansions of the functions $(z/2)^{-a}\mathfrak{F}_{k+2m}(z)$ all converge absolutely for all finite values of z, and (7) converges uniformly in the circle $|z| = R'$, we may sum by columns the repeated series obtained by replacing each of the functions $(z/2)^{-a}\mathfrak{F}_{k+2m}(z)$ in (7) by its power series expansion,* and this sum is $(z/2)^k$, as is seen from the relations (4). Hence the series in (3) represents the function $(z/2)^{a+k}$.

* See Osgood, *Lehrbuch der Funktionentheorie*, Vol. I, 4th Ed., p. 343.

III. Expansion of $f(z)$ in Terms of the Functions $F_m(z-a)$ and of the Functions $B_m(z-a)$.

6. *Expansion of $z^a/(t-z)$ in terms of $\mathfrak{F}_m(z)$.* In the series

$$z^a/(t-z) = \sum_{k=0}^{\infty} z^{a+k}/t^{k+1}, \tag{19}$$

which is valid when $|t| > |z|$, substitute the series expansions for z^{a+k} obtained from (3). This gives

$$z^a/(t-z) = \sum_{k=0}^{\infty} 2^{a+k}/t^{k+1} \sum_{m=0}^{\infty} c_{k,m}\, \mathfrak{F}_{k+2m}(z). \tag{20}$$

Rearrange this repeated series, assuming temporarily that this rearrangement is permissible, into the simple series

$$z^a/(t-z) = \sum_{m=0}^{\infty} B_m(t)\, \mathfrak{F}_m(z), \tag{21}$$

where

$$B_m(t) = \sum_{r=0}^{\leq m/2} c_{m-2r,r}\, 2^{a+m-2r}/t^{m+1-2r},$$

the sum terminating with m 2 if m is even, and with $(m-1)/2$ if m is odd.

Since the absolute convergence of a repeated series implies the absolute convergence of the corresponding double series,* the above rearrangement to obtain (21) is permissible provided the repeated series in (20) is absolutely convergent. To establish this fact we first observe that the repeated series in (20) is dominated, term by term, by the series

$$\sum_{k=0}^{\infty} |z^{a+k}|/|t|^{k+1} \sum_{m=0}^{\infty} P_{k,m}(a')\, |z/2|^{2m} e^{|z/2|^2} E_{k+2m}. \tag{22}$$

If we make use of the relations (12) we are able, by arguments entirely analogous to those employed in Art. 5, to prove that

$$\sum_{m=0}^{\infty} P_{k,m}(a')\, |z/2|^{2m} e^{|z/2|^2} E_{k+2m} \tag{23}$$

is convergent for every finite value of z and k. Also, by the aid of (13) we may prove in a similar manner that

$$\sum_{m=0}^{\infty} P_{k,m}(\alpha')\, |z/2|^{2m} e^{|z/2|^2} E_{k+2m} \tag{24}$$

is convergent for every finite value of z and for every $k > k_1$ where

* Bromwich, "Theory of Infinite Series," p. 78.

$k_1 > |\mu_i|/\theta_i$ for $i = 1, 2, \cdots, n$. (This restriction is placed upon k to insure that all the quantities $P_{k,m}(\alpha')$ shall exist.) For fixed $k \geq k_1$ we see from (14) that the series (23) is dominated, term by term, by the series (24). Denote by $S_k(z)$ the sum of (23), and by $\sigma_k(z)$ the sum of (24), when $k > k_1$. Now, if R is any constant and $|z| \leq R$, then $\sigma_k(z)$ is bounded both as to z and k. For, in the series (24) each term is a monotone decreasing function of k when $k \geq k_1$. Hence, if z and k are in the given regions, $\sigma_{k_1}(R) \geq \sigma_k(z)$. Clearly, if $0 \leq k < k_1$, $S_k(z)$ has a bound, since $S_k(z)$ exists for each value of k, and the number of S's is finite. Let this bound be M_1, and let M be the greater of the two quantities $\sigma_{k_1}(R)$ and M_1. Then we have

$$(25) \qquad \sum_{k=0}^{\infty} |z^{\alpha+k}|/|t|^{k+1} \sum_{m=0}^{\infty} |c_{k,m}| \cdot |\mathfrak{F}_{k+2m}(z)| \leq M \sum_{k=0}^{\infty} |z^{\alpha+k}|/|t|^{k+1},$$

which converges absolutely when $|z| < |t|$. Whence the rearrangement in obtaining (21) is justified.

The uniformity of convergence of the series $\sum_{m=0}^{\infty} B_m(t) z^{-\alpha} \mathfrak{F}_m(z)$ with regard to both z and t throughout the regions $|t| \geq R$, $|z| \leq R'$, where $R > R'$, is readily established. For, when these inequalities are satisfied this series is dominated, term by term, by a series of positive quantities independent of t and z, and the uniformity of convergence follows from the test of Weierstrass. We may state this result as

THEOREM I. *Let z and t be two complex variables such that $|z| \leq R'$ and $|t| \geq R$, where $R > R'$; then the expansion*

$$(26) \qquad 1/(t-z) = \sum_{m=0}^{\infty} B_m(t) F_m(z),$$

where $F_m(z) = z^{-\alpha}\mathfrak{F}_m(z)$, is valid. If t be a variable point on or outside the circle $|z| = R'$, and z a variable point on or within the circle $|z| = R'$, then the expansion converges uniformly with regard to t and z.

7. *Expansion of $z^{\alpha}f(z)$ in terms of the functions $\mathfrak{F}_m(z)$.* Now, by the aid of Cauchy's integral formula and the expansion (26) we are able to obtain an expansion of a function which is analytic in the neighborhood of the origin in terms of the functions $F_m(z)$. For we have

$$(27) \qquad f(z) = (1/2\pi i) \int_C \frac{f(t)\,dt}{(t-z)}$$

$$= (1/2\pi i) \int_C \{ \sum_{m=0}^{\infty} B_m(t)\, F_m(z) \}\, f(t)\, dt$$

$$= \sum_{m=0}^{\infty} a_m F_m(z),$$

where

$$a_m = (1/2\pi i) \int_C B_m(t)\, f(t)\, dt,$$

the path of integration being the circle $|z| = R$, within and upon which $f(z)$ is single-valued and analytic. This expansion converges uniformly with regard to z when $|z| \leq R'$ and $R > R'$. Hence the expansion

$$(28) \qquad z^a f(z) = \sum_{m=0}^{\infty} a_m \mathfrak{F}_m(z),$$

is valid in the same region. We may state this result as

THEOREM II. *Let $f(z)$ be a function of z which is analytic and single-valued within and upon the circle C defined by the relation $|z| = R$: then the expansion*

$$(29) \qquad z^a f(z) = \sum_{m=0}^{\infty} a_m \mathfrak{F}_m(z),$$

where

$$a_m = (1/2\pi i) \int_C B_m(t)\, f(t)\, dt,$$

is valid when z is within the circle $|z| = R$. If $R > R'$ the expansion obtained from (29) by dividing each term by z^a converges uniformly with regard to z when $|z| \leq R'$.

If we let $g(z) = 1/(t-z)$ and form from (21) $g(0), g'(0), g''(0), \cdots$, where the differentiation is with respect to z, it is clear that the polynominals $B_m(t)$ are uniquely determined.* And in a similar manner the expansion (27) is shown to be unique.

8. *Expansion of $f(z)$ in terms of $B_m(z)$.* Again if z and t be two complex variables which satisfy the relations $|z| \geq r$, $|t| < r'$, and $r > r'$;

* To be exact this proves that all expansions of $1/(t-z)$ in terms of the functions $F_m(z)$ which are differentiable, term by term, are identical. This is sufficient for our purpose. It can be proved, however, that *any* expansion of $1/(t-z)$ in terms of $F_m(z)$ which is convergent has a region of uniform convergence, and therefore is identical with (21).

then, by merely interchanging the roles of t and z in Art. 6 we obtain the expansion

$$1/(z-t) = \sum_{m=0}^{\infty} F_m(t)\, B_m(z). \tag{30}$$

If t be a variable point on or within the circle C defined by the relation $|t| = r'$, and z is any point on or outside the circle $|z| = r$, then the expansion (30) converges uniformly with regard to t and z. If $f(z)$ be any function of z which is analytic and single-valued upon and outside the circle C it can be shown, by arguments analogous to those used in Art. 7, to have the expansion

$$f(z) = \sum_{m=0}^{\infty} a'_m\, B_m(z), \quad \text{where} \quad a'_m = (1/2\pi i) \int_C F_m(t)\, f(t)\, dt. \tag{31}$$

And this expansion converges uniformly with regard to z provided z is on or outside the circle $|z| = r$.

9. *Expansion of $f(z)$ in terms of the functions $F_m(z-a)$ and $B_m(z-a)$.* If in the expansion (26) we replace t and z by $t-a$ and $z-a$, respectively, where a is any complex quantity, we obtain the expansion

$$1/(t-z) = \sum_{m=0}^{\infty} B_m(t-a)\, F_m(z-a), \tag{32}$$

which is valid when $|t-a| \geq R$, $|z-a| \leq R'$, and $R > R'$. This expansion converges uniformly with regard to t and z when these inequalities are satisfied.

In a similar manner we obtain from the expansion (30),

$$1/(z-t) = \sum_{m=0}^{\infty} F_m(t-a)\, B_m(z-a), \tag{33}$$

which is valid when $|t-a| \leq r'$, $|z-a| \geq r$, and $r > r'$. And likewise this expansion converges uniformly with respect to t and z when these inequalities hold.

If $f(z)$ is analytic and single-valued within and upon the circle C defined by the relation $|z-a| = R$, then the expansion

$$f(z) = \sum_{m=0}^{\infty} a_m\, F_m(z-a), \quad \text{where} \quad a_m = (1/2\pi i) \int_C B_m(t-a) f(t)\, dt, \tag{34}$$

is valid; and, if z remains in the circle $|z-a| = R'$, where $R > R'$, this expansion converges uniformly with respect to z.

And, if $f(z)$ is analytic and single-valued upon and outside the circle C', defined by the relation $|z-a|=r'$, then the expansion

$$(35)\quad f(z)=\sum_{m=0}^{\infty} a'_m B_m(z-a), \text{ where } a'_m=(1/2\pi i)\int_{C'} F_m(t-a)f(t)\,dt,$$

is valid; and, if z remains on or outside the circle $|z-a|=r$, where $r>r'$, this expansion converges uniformly with respect to z. When $a=0$ the expansions (34) and (35) reduce to (27) and (31), respectively.

IV. Relation to the Taylor and Laurent Expansions.

10. If the Taylor's expansion of $f(z)$ is $f(z)=\sum_{m=0}^{\infty} b_m(z-a)^m$, while the expansion (34) is

$$f(z)=\sum_{m=0}^{\infty} a_m F_m(z-a),$$

it is possible to obtain a relation between the coefficients b_m and a_m in these two expansions. For

$$(36)\quad \begin{aligned} a_m &= (1/2\pi i)\int_C B_m(t-a)\,f(t)\,dt \\ &= (1/2\pi i)\int_C B_m(t-a)\,\{\sum_{s=0}^{\infty} b_s(t-a)^s\}\,dt \\ &= \sum_{s=0}^{\infty} b_s\,(1/2\pi i)\int_C (t-a)^s \sum_{r=0}^{\leq m/2} c_{m-2r,r}\,\frac{2^{a+m-2r}}{(t-a)^{m-1-2r}}\,dt \\ &= \sum_{r=0}^{\leq m/2} 2^{a+m-2r}\,c_{m-2r,r}\,b_{m-2r}. \end{aligned}$$

By means of these relations we are able to readily construct the expansion (29) when the Taylor's expansion for the given function $f(z)$ is known.

Next, let $f(z)$ be an analytic and single-valued function of z in the ring-shaped region S, bounded by the two circles C_2 and C_1 defined by the relations $|z-a|=r$ and $|z-a|=R$, respectively, where $R>r$. Let z be a fixed point in the region S. Then we have from Cauchy's integral formula

$$(37)\quad f(z)=(1/2\pi i)\int_{C_1}\frac{f(t)\,dt}{t-z}+\frac{1}{2\pi i}\int_{C_2}\frac{f(t)\,dt}{t-z},$$

the two paths of integration being taken in a counter-clockwise direction. If we apply the expansions (32) and (33) to the first and second integrals, respectively, we have

$$(38)\qquad f(z) = (1/2\pi i)\int_{C_1} \sum_{m=0}^{\infty} B_m(t-a)F_m(z-a)f(t)\,dt + (1/2\pi i)\int_{C_2} \sum_{m=0}^{\infty} F_m(t-a)\,B_m(z-a)f(t)\,dt = \sum_{m=0}^{\infty} a_m F_m(z-a) + \sum_{m=0}^{\infty} a'_m B_m(z-a),$$

where

$$a_m = (1/2\pi i)\int_{C_1} B_m(t-a)f(t)\,dt \quad \text{and} \quad a'_m = (1/2\pi i)\int_{C_2} F_m(t-a)f(t)\,dt,$$

and this expansion is valid when z is in S. This result may be stated as

THEOREM III. *Let $f(z)$ be a function of z, which is analytic and single-valued in the region defined by the inequalities $0 < r \leq |z-a| \leq R$, and let C_1 and C_2 denote the circles $|z-a| = R$ and $|z-a| = r$, respectively. Then, the expansion*

$$(39)\qquad f(z) = \sum_{m=0}^{\infty} a_m F_m(z-a) + \sum_{m=0}^{\infty} a'_m B_m(z-a),$$

where

$$a_m = (1/2\pi i)\int_{C_1} B_m(t-a)f(t)\,dt$$

and

$$a'_m = (1/2\pi i)\int_{C_2} F_m(t-a)f(t)\,dt,$$

is valid and converges uniformly with regard to z when $r' \leq |z-a| \leq R'$, and $R > R' > r' > r > 0$.

If the Laurent expansion of $f(z)$ in the region $r' \leq |z-a| \leq R'$ is

$$(40)\qquad f(z) = \sum_{m=0}^{\infty} b_m(z-a)^m + \sum_{m=1}^{\infty} b'_m(z-a)^{-m},$$

we are able to obtain relations between the coefficients a_m, a'_m and the coefficients b_m, b'_m in the expansion (39). In fact the value of a_m is expressed in terms of b_m by a relation notationally equivalent to the relation (36) already obtained, and from (40) we have

$$(41)\qquad a'_m = (1/2\pi i)\int_C F_m(t-a)f(t)\,dt = (1/2\pi i)\int_C F_m(t-a)\left\{\sum_{s=1}^{\infty} b'_s(t-a)^{-s}\right\}dt = \sum_{s=1}^{\infty} b'_s\,(1/2\pi i)\int_C (t-a)^{-s}F_m(t-a)\,dt = \sum_{r=0}^{\infty} \frac{(-1)^r a_{m,r}}{2^{m+2r}}\, b'_{m+1+2r}.$$

By means of these relations it is possible to construct the Neumann series expansion of a given function when the Taylor or Laurent series for the given function is known.*

V. Orthogonality Properties.

11. We shall next obtain the following integral formulae:

$$\int_C F_m(t-a)\, F_r(t-a)\, dt = 0, \text{ for all values of } m \text{ and } r,$$

$$\int_C B_m(t-a)\, B_r(t-a)\, dt = 0, \text{ for all values of } m \text{ and } r,$$

(42)

$$\int_C F_m(t-a)\, B_m(t-a)\, dt = 2\pi i k, \text{ for } c \text{ enclosing the point } z = a,$$

$$= 0, \text{ for } c \text{ not enclosing the point } z = a,$$

$$\int_C F_m(t-a)\, B_r(t-a)\, dt = 0, \text{ for } m \neq r,$$

where the contour C, in each case, is any closed curve not passing through the point $z = a$. Denote by S the region enclosed by the contour C.

The first formula is true because the integrand is analytic throughout S and along its boundary, and hence Cauchy's theorem applies.

The second formula is seen to be true, by observing that the residue of $B_m(t-a)B_r(t-a)$ at $t = a$ is zero, and that it is analytic at all other points in S.

The third formula follows from the fact that the integrand has a pole of the first order at $t = a$ and at this point the residue is unity. We see that the integral is $2\pi i\kappa$ or 0 according as C does or does not enclose the point $z = a$ where κ is the excess of the number of positive over the number of negative circuits of C about the point $z = a$.

To establish the fourth formula we separate the argument into two cases, the one for which $m > r$ and the other for which $m < r$. If $m > r$ the residue of the integrand at $t = a$ is readily seen to be zero. Hence the formula is true for this case. If $m < r$ we may write $m = r - p$ where p is of the set $1, 2, 3, \cdots, r$. If p is odd the residue at $t = a$ is seen to be zero. If p is even denote it by $2s$. Then

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* Cf. Neumann, *Theorie der Bessel'schen Funktionen* (Leipzig, 1867), pp. 33-35; also Watson, *l. c.*, pp. 523-524.

The residue of $F_m(t-a)B_{m+2s}(t-a)$ at $t=a$ is

$$a_{m,0}\,c_{m,s} - a_{m,1}\,c_{m+2,s-1} + \cdots + (-1)^s\, a_{m,s}\,c_{r,0},$$

which may be written in the form *

$$\begin{vmatrix} a_{m,0} & a_{m,0} & 0 & 0 & \cdots & 0 \\ a_{m,1} & a_{m,1} & a_{m+2,0} & 0 & \cdots & \\ a_{m,2} & a_{m,2} & a_{m+2,1} & a_{m+4,0} & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{m,s-1} & a_{m,s-1} & a_{m+2,s-2} & a_{m+4,s-3} & \cdots & a_{m+2s-2,0} \\ a_{m,s} & a_{m,s} & a_{m+2,s-1} & a_{m+4,s-2} & \cdots & a_{m+2s-2,1} \end{vmatrix}$$

$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$a_{m,0} \quad a_{m+2,0} \quad \cdots \quad a_{m+2s,0}$$

Since the first two columns of the determinant are identical the residue at $t=a$ is zero, and the fourth formula is verified. The formulae (42) afford a convenient means for determining the coefficients in some of the expansions which we have developed, provided the expansions are known to exist. For example, if we wish to determine the coefficients in the expansion (27) we assume an expansion of the form

$$f(z) = \sum_{m=0}^{\infty} a_m F_m(z),$$

multiply both members by $B_m(z)$, and integrate along the circle $|z| = R$, within and upon which $f(z)$ is assumed analytic and single-valued. Making use of the third and fourth formulae of (42) we obtain for a_m the value given in (27). It is clear that the coefficients in other of our expansions may be similarly determined.

VI. Expansions for which the Sum of the θ's is 2.

12. Thus far in our development we have assumed that the sum of the θ's in the functions $\mathfrak{F}_m(z)$ is unity. In fact that assumption was essential to the argument which was employed. For, if the sum of the θ's has *any* value other than unity, and k be not an even integer, we are led to the consideration of a system of inconsistent equations for determining the coefficients

* This statement is verified if we expand the determinant by minors with respect to the elements of the first column, and observe that the minor obtained from the determinant by striking out the first $(r+1)$ rows and the first $(r+1)$ columns is given by the relation

$$\Delta'_{m+2r,\,s-r} = c_{m+2r,\,s-r}\; a_{m+2r,\,0}\; a_{m+2r+2,\,0} \cdots a_{m+2s,\,0}.$$

in the formal expansion of $(z/2)^{a+k}$ in terms of the functions $\mathfrak{F}_m(z)$. However, if k be an even integer, and the sum of the θ's is 2, then we may obtain a system of equations which is consistent, and then it is possible to get an expansion for $(z/2)^{a+2r}$ in a series of the functions $\mathfrak{F}_{r+m}(z)$, in which the sum of the θ's is 2, and r is any non-negative integer. This suggests the possibility of the expansion of even analytic functions in terms of $z^{-a}\mathfrak{F}_m(z)$, in which the sum of the θ's is 2. Therefore, we next obtain such an expansion, then a similar expansion for odd functions, and finally by combining these two expansions get an expansion for any function which is analytic and single-valued in the neighborhood of the origin in terms of the functions $z^{-a}\mathfrak{F}_m(z)$ and $z^{-a+1}\mathfrak{F}_m(z)$ when the sum of the θ's is 2. In this article the quantities $a_{m,s}$ and $\mathfrak{F}_m(z)$ have precisely the same form as in the previous articles, but we shall understand that throughout this article we are carrying the assumption that the sum of the θ's is 2. The argument for this case is very similar to that used in the first part of this paper, and, consequently, much of the details may be omitted.

Assume an expansion of the form

$$(43) \qquad (z/2)^{a+2r} = \sum_{m=0}^{\infty} C_{r,m}\,\mathfrak{F}_{r+m}(z)$$

and substitute for each of the functions $\mathfrak{F}_{r+m}(z)$ the appropriate series expansion, and equate coefficients of like powers of $z/2$. From the resulting recurrence relations the quantities $C_{r,m}$ may be determined, and are seen to be

$$C_{r,m} = \frac{\begin{vmatrix} a_{r,1} & a_{r+1,0} & 0 & 0 & 0 \cdots 0 \\ a_{r,2} & a_{r+1,1} & a_{r+2,0} & 0 & 0 \cdots \\ a_{r,3} & a_{r+1,2} & a_{r+2,1} & a_{r+3,0} & 0 \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{r,m-1} & a_{r+1,m-2} & a_{r+2,m-3} & a_{r+3,m-4} & \cdots a_{r+m-1,0} \\ a_{r,m} & a_{r+1,m-1} & a_{r+2,m-2} & a_{r+3,m-3} & \cdots a_{r+m-1,1} \end{vmatrix}}{a_{r,0} \quad a_{r+1,0} \quad \cdots \quad a_{r+m,0}}$$

The convergence of the series in (43) and its representation of the function $z/2^{a+2r}$ is proved by arguments analogous to those used in Arts. 2 to 5.

For each of the powers of $z/2$ in the expansion

$$z^a/(l^2 - z^2) = \sum_{r=0}^{\infty} (z^{a+2r}/l^{2r+2}),$$

substitute the corresponding expansion from (43) and rearrange the repeated series into a simple series. The resulting formal expansion,

$$(44) \qquad z^a/(t^2 - z^2) = \sum_{m=0}^{\infty} B_m(t)\, \mathfrak{F}_m(z),$$

where $$B_m(t) = \sum_{s=0}^{m} c_{s,m-s}\, (2^{a+2s}/t^{2s+2}),$$

is shown to be valid by the method used in Art. 6.

Now, let $f(z)$ be an even function of the complex variable z, single-valued and analytic, within and upon the circle C defined by the relation $|z| = R$. Then by the special form of Cauchy's integral formula employed by Neumann,* namely,

$$f(z) = (1/2\pi i) \int_C \frac{f(t)\, t\, dt}{(t^2 - z^2)} \quad ,$$

and the expansion (44) we have

$$(45) \qquad z^a f(z) = (1/2\pi i) \int_C \frac{z^a f(t)\, t\, dt}{t^2 - z^2} = \sum_{m=0}^{\infty} c_m\, \mathfrak{F}_m(z),$$

where $$c_m = (1/2\pi i) \int_C B_m(t)\, f(t)\, t\, dt.$$

This expansion is valid when z is inside the circle $|z| = R$, and converges uniformly with regard to z when $|z| \leq R'$, where $R > R'$. Since any odd function of z, which is single-valued and analytic at $z = 0$, may be expressed as z multiplied by an even analytic function, we are able to get the expansion for odd analytic functions in a series very similar to (45). For, let $\phi(z) = z f(z)$, where $\phi(z)$ is an odd analytic function of z. Then $f(z)$ is even, and has the expansion

$$f(z) = \sum_{m=0}^{\infty} z^{-a} C_m\, \mathfrak{F}_m(z).$$

Consequently,

$$\phi(z) = \sum_{m=0}^{\infty} z^{-a+1} C_m\, \mathfrak{F}_m(z), \quad \text{where} \quad C_m = (1/2\pi i) \int_C B_m(t)\, \phi(t)\, dt.$$

We may state these results as

THEOREM IV. *Let $f(z)$ and $\phi(z)$ be an even and an odd function of z, respectively, each analytic and single-valued in the region $|z| = R$, and along its contour C. Then the expansions*

$$(46) \qquad f(z) = \sum_{m=0}^{\infty} z^{-a} c_m\, \mathfrak{F}_m(z), \quad \textit{where} \quad c_m = (1/2\pi i) \int_C B_m(t)\, f(t)\, t\, dt,$$

$$(47) \qquad \phi(z) = \sum_{m=0}^{\infty} z^{-a+1} C_m \mathfrak{F}_m(z), \quad \textit{where} \quad C_m = (1/2\pi i) \int_C B_m(t)\, \phi(t)\, dt,$$

* *Mathematische Annalen*, Vol. 3, p. 594.

are valid when z is within the circle $|z| = R$, and these expansions both converge uniformly with regard to z when $|z| \leq R'$, where $R > R'$.

Let $f(z)$ be any function of z which is analytic and single-valued in the region $|z| = R$ and upon its contour C. And, let $f(z) = \phi_1(z) + \phi_2(z)$, where $\phi_1(z)$ is an odd function, and $\phi_2(z)$ is an even function, each of which is analytic and single-valued in the circle $|z| = R$. Then by the expansions (46) and (47) we have at once the expansion

$$f(z) = \sum_{m=0}^{\infty} z^{-a+1} c_m \mathfrak{F}_m(z) + \sum_{m=0}^{\infty} z^{-a} c'_m \mathfrak{F}_m(z), \tag{48}$$

where $$c_m = (1/2\pi i) \int_C B_m(t)\, \phi_1(t)\, dt,$$

and $$c'_m = (1/2\pi i) \int_C B_m(t)\, \phi_2(t)\, t\, dt.$$

This expansion is valid when z is within the circle $|z| = R$, and the convergence is uniform with regard to z when $|z| \leq R'$, where $R > R'$. Furthermore, the uniqueness of the expansions (44) and (45) can be established in a manner analogous to that used in Art. 7.

VII. Special Cases.

13. In order to make explicit the sense in which these expansions are generalizations of known results it seems desirable to call attention to some of the special cases which have been treated. It will then become apparent what generalizations have been made.

When $n = 1$, $\mu_1 = 0$, and $\theta_1 = 1$ the expansion (29) reduces to that obtained by Neumann,* namely

$$f(z) = \sum_{m=0}^{\infty} a_m J_m(z),$$

where $$a_m = (\epsilon_m/2\pi i) \int_C O_m(t)\, f(t)\, dt, \qquad \epsilon_m = 2 \text{ if } m \geq 1, \quad \epsilon_0 = 1,$$

and $$\begin{cases} O_m(t) = (1/4) \sum_{r=0}^{\leq m/2} \dfrac{m(m-r-1)!}{r!(t/2)^{m+1-2r}} & \text{for } m \geq 1. \\ O_0(t) = 1/t. \end{cases}$$

For, since the expansion (21) is unique it follows that $\epsilon_m O_m(t) = B_m(t)$ when $\mu_1 = 0$ and $\theta_1 = 1$, n now having the value unity.

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* *Theorie der Bessel'schen Funktionen*, pp. 33-35; Watson, *loc. cit.*, p. 523.

When $n = 2$, $\theta_1 = \theta_2 = 1$, and $\mu_1 = \mu_2 = 0$ we see from the uniqueness of the expansion (44) that the expansion of even functions given in (46) reduces to that obtained by Neumann * for even functions in terms of squares of Bessel Functions, namely,

$$f(z) = \sum_{m=0}^{\infty} a'_m [J_m(z)]^2, \quad \text{where } a'_m = (\epsilon_m/2\pi i) \int_C \Omega_m(t) f(t)\, t\, dt,$$

and

$$\begin{cases} \Omega_m(t) = (1/4) \sum_{s=0}^{m} \dfrac{m(m+s-1)!\,(s!)^2}{(m-s)!\,(2s)!\,(t/2)^{2s+2}} & \text{for } m \geq 1, \\ \Omega_0(t) = 1/t^2. \end{cases}$$

And we have $\epsilon_m \Omega_m(t) = B_m(t)$, where $\theta_1 = \theta_2 = 1$ and $\mu_1 = \mu_2 = 0$, n now having the value 2.

When $n = 1$, $\mu_1 = \nu$, and $\theta_1 = 1$ the expansion (29) reduces to the following expansion, due to Gegenbauer,†

$$z^\nu f(z) = \sum_{m=0}^{\infty} a_m J_{\nu+m}(z), \text{ where } a_m = (1/2\pi i) \int_C A_{m,\nu}(t) f(t)\, dt,$$

and

$$A_{m,\nu}(t) = \frac{2^{\nu+m}(\nu+m)}{t^{m+1}} \sum_{s=0}^{\leq m/2} \frac{\Gamma(\nu+m-s)}{s!} (t/2)^{2s}.$$

From the uniqueness of the expansion (21) we have $A_{m,\nu}(t) = B_m(t)$ when $\mu_1 = \nu$ and $\theta_1 = 1$, n now having the value 1.

When $n = 2$, $\mu_1 = \mu$, $\mu_2 = \nu$ and $\theta_1 = \theta_2 = 1/2$ we obtain from (29) another expansion, due also to Gegenbauer,‡

$$z^{\mu+\nu} f(z) = \sum_{m=0}^{\infty} a_m J_{\mu+(m/2)}(z)\, J_{\nu+(m/2)}(z),$$

$$a_m = (1/2\pi i) \int_C B_{m,\mu,\nu}(t) f(t)\, dt,$$

and

$$B_{m,\mu,\nu}(t) = \frac{2^{\mu+\nu+m}(\mu+\nu+m)}{t^{m+1}} \sum_{s=0}^{\leq m/2} \frac{\Gamma[\mu+(m/2)-s+1]\,\Gamma[\nu+(m/2)-s+1]\,\Gamma(\mu+\nu+m-s)}{s!\,(\mu+\nu+m-2s+1)} (t/2)^{2s}.$$

And, likewise, $B_{m,\mu,\nu}(t) = B_m(t)$ when $\mu_1 = \mu$, $\mu_2 = \nu$ and $\theta_1 = \theta_2 = 1/2$, n now having the value 2.

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* *Mathematische Annalen*, Vol. 3 (1871), p. 599; Watson, *loc. cit.*, p. 525.

† *Wiener Sitzungsberichte*, 74, (2), (1877), pp. 124-130; Watson, *loc. cit.*, p. 525.

‡ *Wiener Sitzungsberichte*, 75, (2), (1877), pp. 218-222; Watson, *loc. cit.*, p. 525.

VIII. Expansions of Analytic Functions of Several Variables.

14. Much of the foregoing expansion theory which has been developed for functions of a single variable is capable of extension to functions of several variables. We shall next indicate briefly the argument for one of the expansions which can be obtained.

Since $\mathfrak{F}_m(z)$ is a function of the quantities μ_i and θ_i, we introduce the following notation:

$$\mathfrak{F}_m^{(j)}(z) = J_{\mu_{j1}+\theta_{j1}m}(z)\, J_{\mu_{j2}+\theta_{j2}m}(z) \cdots J_{\mu_{j_{\nu_j}}+\theta_{j\nu_j}m}(z),$$

$$B_m^{(j)}(t) = \sum_{r=0}^{\leq m/2} c_{m-2r,r}\, \frac{2^{\alpha_j+m-2r}}{t^{m+1-2r}},$$

$$\alpha_j = \mu_{j1} + \mu_{j2} + \cdots + \mu_{j\nu_j},$$

with the restriction that $\mu_{ji} + \theta_{ji}r$ be not a negative integer for any i or j occurring in the expansion, and for $r = 0, 1, 2, \cdots$.

For simplicity in argument the expansion is developed for the case of two variables. The extension is obvious.

Let z_1, t_1, and z_2, t_2 be two pairs of complex quantities in the z_1 and z_2 planes, respectively, such that

$$(49)\quad \begin{cases} |z_1| \leq r_1, & |t_1| \geq R_1, & r_1 < R_1, \\ |z_2| \leq r_2, & |t_2| \geq R_2, & r_2 < R_2; \end{cases}$$

then, by Theorem I the expansions

$$(50)\quad \begin{cases} 1/(t_1 - z_1) = \sum_{m_1=0}^{\infty} B_{m_1}^{(1)}(t_1)\, F_{m_1}^{(1)}(z_1), \\ 1/(t_2 - z_2) = \sum_{m_2=0}^{\infty} B_{m_2}^{(2)}(t_2)\, F_{m_2}^{(2)}(z_2), \end{cases}$$

where $$F_{m_i}^{(r)}(z_i) = z^{-\alpha_r}\mathfrak{F}_{m_i}^{(r)}(z_i),$$

are valid. If t_1 and t_2 are variable points on or outside the circles C_1 and C_2, respectively, defined by the relations $|z_1| = R_1$, $|z_2| = R_2$, and if z_1 and z_2 are any variable points within or upon the circles $|z_1| = r_1$ and $|z_2| = r_2$, respectively, then the expansions (50) converge uniformly with regard to t_1, z_1 and t_2, z_2, respectively. Since both expansions converge uniformly when conditions (49) are fulfilled and are dominated by convergent series of constants, then under the same conditions the double series in the second member of the equation

$$(51)\qquad 1/(t_1 - z_1)\cdot 1/(t_2 - z_2) = \sum_{m_1, m_2=0}^{\infty} B_{m_1}^{(1)}(t_1)\, B_{m_2}^{(2)}(t_2)\, F_{m_1}^{(1)}(z_1)\, F_{m_2}^{(2)}(z_2)$$

converges uniformly.

Now let $f(z_1, z_2)$ be an arbitrary function of the two independent complex variables z_1 and z_2, which is analytic and single-valued in z_1 and z_2 separately when z_1 and z_2 are, respectively, in the regions bounded by the contours C_1 and C_2; then, by Cauchy's generalized integral formula, we have

$$\begin{aligned} f(z_1, z_2) &= (1/2\pi i)^2 \int_{C_1} \int_{C_2} \frac{f(t_1, t_2)\, dt_1\, dt_2}{(t_1 - z_1)(t_2 - z_2)} \\ &= (1/2\pi i)^2 \int_{C_1} \int_{C_2} \Big\{ \sum_{m_1, m_2=0}^{\infty} B_{m_1}^{(1)}(t_1)\, B_{m_2}^{(2)}(t_2) \\ &\qquad F_{m_1}^{(1)}(z_1)\, F_{m_2}^{(2)}(z_2) \Big\} f(t_1, t_2)\, dt_1\, dt_2 \\ &= \sum_{m_1, m_2=0}^{\infty} A_{m_1, m_2} F_{m_1}^{(1)}(z_1)\, F_{m_2}^{(2)}(z_2), \end{aligned}$$

where $A_{m_1, m_2} = 1/(2\pi i)^2 \int_{C_1} \int_{C_2} B_{m_1}^{(1)}(t_1)\, B_{m_2}^{(2)}(t_2)\, f(t_1, t_2)\, dt_1\, dt_2$.

This expansion is valid and converges uniformly with regard to z_1 and z_2 when z_1 and z_2 are, respectively, in the circles $|z_1| = r_1$ and $|z_2| = r_2$. This result may be stated for ν variables as

THEOREM V. *Let $f(z_1, z_2, \cdots, z_\nu)$ be a function of the complex quantities $z_1, z_2, \cdots, z_\nu$, which is analytic and single-valued when $z_1, z_2, \cdots, z_\nu$ are, respectively, in the regions $|z_1| \leqq R_1$, $|z_2| \leqq R_2$, $\cdots$, $|z_\nu| \leqq R_\nu$, whose contours are, respectively, $C_1, C_2, \cdots, C_\nu$, then the expansion*

$$f(z_1, z_2, \cdots, z_\nu) = \sum_{m_1, m_2, \ldots, m_\nu=0}^{\infty} A_{m_1, m_2, \ldots m_\nu}\, F_{m_1}^{(1)}(z_1)\, F_{m_2}^{(2)}(z_2) \cdots F_{m_\nu}^{(\nu)}(z_\nu),$$

where

$$\begin{aligned} A_{m_1, m_2, \ldots m_\nu} = 1/(2\pi i)^\nu \int_{C_1} \int_{C_2} \cdots \int_{C_\nu} & B_{m_1}^{(1)}(t_1)\, B_{m_2}^{(2)}(t_2) \\ & \cdots B_{m_\nu}^{(\nu)}(t_\nu) f(t_1, t_2, \cdots, t_\nu)\, dt_1\, dt_2 \cdots dt_\nu, \end{aligned}$$

is valid and converges uniformly when $z_1, z_2, \cdots, z_\nu$ are, respectively, in the circles $|z_1| = r_1, |z_2| = r_2, \cdots, |z_\nu| = r_\nu$, where $r_i < R_i$, $i = 1, 2, \cdots, \nu$.

It is now obvious that expansions of functions of several variables, analogous to the expansions (31), (34), (35) and (39) exist.

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VITA.

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